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Automorphisms of tube domains

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A tube domain T_Ω in $\mathbb{C}^n$ is a domain in $\mathbb{C}^n$ given by $T_\Omega = \mathbb{R}^n + \sqrt{-1}\Omega$, where Ω is a domain in $\mathbb{R}^n$ and is called the base of T_Ω. When the additive group $\Sigma := \mathbb{R}^n$ is viewed as the group of real translations of $\mathbb{C}^n$, the group Σ acts naturally on T_Ω. Tube domains in $\mathbb{C}^n$ are characterized as domains in $\mathbb{C}^n$ with the Σ-action.

Proposition. Let $\varphi: T_{\Omega_1} \longrightarrow T_{\Omega_2}$ be a biholomorphic mapping between two tube domains in $\mathbb{C}^n$. If φ is equivariant with respect to the Σ-actions on T_{Ω_1} and T_{Ω_2}, then φ is given by an element of $GL(n, \mathbb{R}) \ltimes \mathbb{C}^n$, where $GL(n, \mathbb{R}) \ltimes \mathbb{C}^n$ denotes the group of all complex affine transformations of $\mathbb{C}^n$ whose linear parts belong to $GL(n, \mathbb{R})$.

This proposition implies that biholomorphic mappings between tube domains with respect to the Σ-actions may be considered as natural isomorphisms in the category of tube domains. We say that two tube domains in $\mathbb{C}^n$ are affinely equivalent if there is a biholomorphic mapping between them given by an element of $GL(n, \mathbb{R}) \ltimes \mathbb{C}^n$.

If the convex hull of the base Ω of a tube domain T_Ω in

$\mathbb{C}^n$ contains no complete straight lines, then T_Ω is biholomorphically equivalent to a bounded domain in $\mathbb{C}^n$ and, by a well-known theorem of H. Cartan [1], the group $\mathrm{Aut}(T_\Omega)$ of all holomorphic automorphisms of T_Ω forms a Lie group with respect to the compact-open topology. The Lie algebra $\mathfrak{g}(T_\Omega)$ of the Lie group $\mathrm{Aut}(T_\Omega)$ can be identified canonically with the finite-dimensional real Lie algebra consisting of all complete holomorpic vector fields on T_Ω. Throughout this article, we deal with tube domains whose bases have the convex hull containing no complete straight lines.

Now, a tube domain whose base is a convex cone is called a Siegel domain of the first kind. This kind of domains cover an important class of complex bounded domains including symmetric domains of tube type. For holomorphic automorphisms and the equivalences of Siegel domains T_Ω of the first kind, the beutiful results are known (cf. Matsushima [2]), and, as a main result, the structure of $\mathfrak{g}(T_\Omega)$ is clarified. On the other hand, various problems, for example, the problem of determining the convex realizations of homogeneous bounded domains, motivate the study of holomorphic automorphisms and the equivalences of tube domains whose bases are not necessarily convex cones. The purpose of this article is to give the structure theorem for the Lie algebra $\mathfrak{g}(T_\Omega)$ when the base Ω of a tube domain T_Ω is an arbitary domain whose convex hull containing no complete straight lines, and to present some applications of our theorem. For brevity, we write $\partial_i = \partial/\partial z_i$.

Theorem ([3]). To each tube domain T_Ω in $\mathbb{C}^n$ whose base Ω has the convex hull containing no complete straight lines, there is associated a tube domain $T_{\tilde{\Omega}}$ which is affinely equivalent to T_Ω such that $\mathfrak{g}(T_{\tilde{\Omega}})$ has the direct sum decomposition

$$\mathfrak{g}(T_{\tilde{\Omega}}) = \mathfrak{p} + \mathfrak{e}$$

for which

$$\mathfrak{p} = \{X \in \mathfrak{g}(T_{\tilde{\Omega}}) \mid X \text{ is a polynomial vector field}\},$$

$$\mathfrak{e} = \sum_{i=1}^{r} \{e^{z_i}(\partial_i + \sum_{j=r+1}^{n} \sqrt{-1}a_i^j \partial_j),\ e^{-z_i}(\partial_i - \sum_{j=r+1}^{n} \sqrt{-1}a_i^j \partial_j)\}_{\mathbb{R}},$$

where r is an integer between 0 and n and a_i^j, $i = 1, \cdots, r$, $j = r + 1, \cdots, n$, are real constants.

Example. (i) Consider the upper half plane $T_{(0,\infty)} = \{z \in \mathbb{C} \mid 0 < \operatorname{Im} z < \infty\}$ in the complex plane. Then we have

$$\mathfrak{g}(T_{(0,\infty)}) = \{\partial,\ z\partial,\ z^2\partial\}_{\mathbb{R}},$$

where $\partial = \partial/\partial z$.

(ii) Consider a strip $T_{(a,b)} = \{z \in \mathbb{C} \mid a < \operatorname{Im} z < b\}$ in the complex plane, where $-\infty < a < b < +\infty$. Then we have

$$\mathfrak{g}(T_{(a,b)}) = \{\partial,\ Ce^{cz}\partial,\ C^{-1}e^{-cz}\partial\}_{\mathbb{R}},$$

where $c = \pi/(b - a)$ and $C = e^{-\sqrt{-1}\pi a/(b - a)}$. In particular, we have

$$\mathfrak{g}(T_{(0,\pi)}) = \langle \partial,\ e^{z}\partial,\ e^{-z}\partial \rangle_{\mathbb{R}} .$$

In our theorem, if, in particular, Ω is a bounded domain in $\mathbb{R}^n$, then we have the following stronger result.

Theorem A ([4]). To each tube domain T_Ω in $\mathbb{C}^n$ whose base Ω is a bounded domain in $\mathbb{R}^n$, there is associated a tube domain $T_{\tilde{\Omega}}$ which is affinely equivalent to T_Ω and has the splitting

$$T_{\tilde{\Omega}} = T_{\tilde{\Omega}'} \times T_{\tilde{\Omega}''}$$

such that:

(i) $T_{\tilde{\Omega}'}$ and $T_{\tilde{\Omega}''}$ are tube domains in $\mathbb{C}^r$ and $\mathbb{C}^{n-r}$, respectively, where r is an integer between 0 and n;

(ii) $T_{\tilde{\Omega}'}$ is given by $T_{\tilde{\Omega}'} = (T_{(0,\pi)})^r$;

(iii) $T_{\tilde{\Omega}''}$ satisfies $\mathrm{Aut}(T_{\tilde{\Omega}''})^\circ = \mathrm{Aff}(T_{\tilde{\Omega}''})^\circ$ and $\mathrm{Aff}(\tilde{\Omega}'') \subset O(n-r)$, where $\mathrm{Aff}(T_{\tilde{\Omega}''})$ denotes the group of complex affine transformations of $\mathbb{C}^{n-r}$ leaving $T_{\tilde{\Omega}''}$ invariant, while $\mathrm{Aff}(\tilde{\Omega}'')$ denotes the group of affine transformations of $\mathbb{R}^{n-r}$ leaving $\tilde{\Omega}''$ invariant, and G° denotes the identity component of G for a Lie group G.

The tube domain $T_{\tilde{\Omega}}$ given in Theorem A is called the normalized form of T_Ω.

Now, the holomorphic equivalence problem for tube domains may be formulated as follows:

Problem. If two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbb{C}^n$ are biholomorphically equivalent, then are they affinely equivalent?

When T_{Ω_1} and T_{Ω_2} are Siegel domains of the first kind, or Ω_1 and Ω_2 are convex cones in $\mathbb{R}^n$, an affirmative answer is given (cf. Matsushima [2]). On the other hand, when Ω_1 and Ω_2 are arbitrary domains in $\mathbb{R}^n$ whose convex hulls contain no complete straight lines, there is a simple counter example. In fact, the tube domains $T_{(0,\pi)}$ and $T_{(0,\infty)}$ in $\mathbb{C}$ are biholomorphically equivalent, but not affinely equivalent. However, as an application of Theorem A, we have the following result.

Theorem B ([4]). If two tube domains T_{Ω_1} and T_{Ω_2} in $\mathbb{C}^n$ whose bases Ω_1 and Ω_2 are bounded domains in $\mathbb{R}^n$ are biholomorphically equivalent, then they are affinely equivalent.

A reasoning in the proof of Theorem B also yields a description of the holomorphic automorphism group of the normalized form of a tube domain with bounded base.

Theorem C ([4]). Let $T_{\tilde{\Omega}}$ be the normalized form of a tube domain in $\mathbb{C}^n$ whose base is a bounded domain in $\mathbb{R}^n$, and let $T_{\tilde{\Omega}} = T_{\tilde{\Omega}'} \times T_{\tilde{\Omega}''}$ be the splitting of $T_{\tilde{\Omega}}$ given in Theorem A. Then $\mathrm{Aut}(T_{\tilde{\Omega}})$ is given by

$$\mathrm{Aut}(T_{\tilde{\Omega}}) = \mathrm{Aut}(T_{\tilde{\Omega}'}) \times \mathrm{Aff}(T_{\tilde{\Omega}''}) .$$

Corollary (cf. Yang [5]). Let T_Ω be a tube domain in $\mathbb{C}^n$ whose base Ω is a bounded domain in $\mathbb{R}^n$. If Ω has C^1-boundary, then $\mathrm{Aut}(T_\Omega) = \mathrm{Aff}(T_\Omega)$.

References

[1] H. Cartan, Sur les groupes de transformations analytiques, Actualites Sci. Ind., Hermann, Paris, 1935.

[2] Y. Matsushima, On tube domains, in Symmetric Spaces, Pure and Appl. Math., Vol. 8, Dekker, New York, 1972, 255-270.

[3] S. Shimizu, Automorphisms of tube domains, preprint.

[4] S. Shimizu, Automorphisms and equivalence of tube domains with bounded base, preprint.

[5] P. Yang, Automorphisms of tube domains, Amer. J. Math. 104 (1982), 1005-1024.